PG – 618

Second Semester M.Sc. Degree Examination, July 2017
(RNS – Repeaters) (2011-12 and Onwards)
MATHEMATICS
M – 201 : Algebra – II

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any five full** questions. Choosing atleast **two** from **each** Part.*
*2) **All** questions carry **equal** marks.*

PART – A

1. a) Let K be an extension of a field F and $a \in K$ be algebraic over F and of degree n. Prove that $[F(a) : F] = n$.

   b) Prove that every finite extension K of a field F is algebraic and may be obtained from F by the adjunction of finitely many algebraic elements.

   c) Let $a = \sqrt{2}$, $b = \sqrt[4]{2}$ in R, where R is an extension of Q. Verify that a + b and ab are algebraic of degree atmost (deg a) (deg b). **(6+6+4)**

2. a) Prove that a polynomial of degree n over a field F can have atmost n roots in any extension field K.

   Is the result true, when extension field K is not a field F ? Justify with an example.

   b) Define the splitting field of a polynomial over a field F.

   Determine the splitting field of

   i) $x^3 - 2$

   ii) $x^2 + x + 1$, over the field Q. **(8+8)**

3. a) Show that the polynomial $f(x) \in F[x]$ has multiple roots if and only if f(x) and $f'(x)$ have a non-trivial common factors.

   b) Prove that a regular pentagon is constructible by using edge and compass.

   c) Show that any field of characteristic zero is perfect field. **(6+5+5)**

P.T.O.

4. a) Define a fixed field. Let G be a subgroup of the group of all automorphisms of a field K. Then show that fixed field of G is a subfield of K.

   b) State and prove the fundamental theorem of Galois theory. **(4+12)**

PART – B

5. a) Let V be a finite – dimensional vector space over F. Prove that T $\in$ A(V) is invertible iff and only iff the constant term of the minimal polynomial for T is not zero.

   b) Give an example to show that ST $\neq$ TS for S, T $\in$ A (V).

   c) Define a rank of a linear transformation T. If V is a finite dimensional vector space over F and S, T $\in$ A (V), then prove that r(ST) $\leq$ min {r (T), r(S)}. **(6+4+6)**

6. a) If T, S $\in$ A(V) and if S is regular, then show that T and STS^{-1} have the same minimal polynomial.

   b) Define a characteristic root of T $\in$ A(V). If $\lambda \in$ F is a characteristic root of T $\in$ A(V), then for any q(x) $\in$ F[x], prove that $q(\lambda)$ is a characteristic root of q(T).

   c) If V is n-dimensional vector space over a field F and if T $\in$ A(V) has the matrix $m_1(T)$ in the basis $\{v_1, v_2, ..., v_n\}$ and the matrix m_2 (T) in the basis $\{w_1, w_2, ..., w_n\}$ of V over F, then prove that there is a matrix C in F_n such that $m_2(T) = C\, m_1(T)\, C^{-1}$. **(5+5+6)**

7. a) If V is n-dimensional vector space over F and if T$\in$A(V) has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F.

   b) Let T $\in$ A(V) and V_1 an n_1-dimensional subspace of V and spanned by $\{v, vT, vT^2, ..., vT^{n-1}\}$, where v $\neq$ 0. If u $\in V_1$ is such that $UT^{n_1-k} = 0, 0 < k \leq n_1$, then show that $u_0T^k = u$ for some $u_0 \in V_1$.

   c) Prove that two nil potent linear transformations are similar iff and only iff they have the same invariants. **(6+4+6)**

8. a) Let V be a finite dimensional complex inner product space. If T $\in$ A(V) is such that $\langle T(v) : v \rangle = 0$ for each v $\in$ V, then prove that T = 0.

   b) If λ is a characteristic root of the normal transformation N and if $vN = \lambda v$, then show that $vN^* = \bar{\lambda} v$.

   c) State and prove the Sylvester's law of inertia for real quadratic form. **(4+4+8)**
